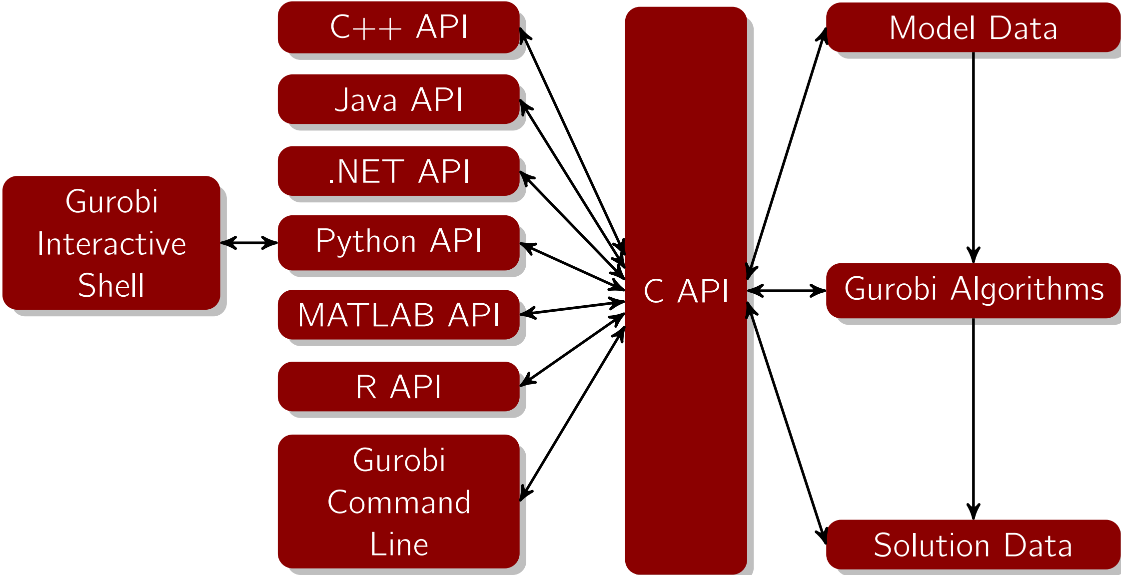
GUROBI
OPTIMIZATION



Gurobi Interactive Shell
C++ API
Java API
.NET API
Python API
MATLAB API
R API
Gurobi Command Line
C API
Model Data
Gurobi Algorithms
Solution Data

$$y = max(x_1, x_2, ..., c)$$

$$y = min(x_1, x_2, ..., c)$$

$$y = |x|$$

$$y = x_1 \wedge x_2 \wedge x_3 ...$$

$$y = x_1 \vee x_2 \vee x_3 ...$$

$$y = 1 \rightarrow a'x \leq b$$

$$y = pwl(x)$$

$$y = p_0 x^d + p_1 x^{d-1} + ... + p_{d-1} x + p_d$$

$$y = e^x$$

$$y = a^x$$

$$y = log_e(x)$$

$$y = log_a(x)$$

$$y = x^a$$

$$y = sin(x)$$

$$y = cos(x)$$

$$y = tan(x)$$

$$r = \max\{x_1, \ldots, x_n, c\}$$

r

$x_1, \ldots, x_n$

c

n

x_j

$$r = \min\{x_1, \ldots, x_n, c\}$$

$$r = \mathrm{abs}\{x\}$$

x

$$r = \text{and}\{x_1, \ldots, x_n\}$$

1

0

$$r = \mathrm{or}\{x_1, \ldots, x_n\}$$

$$z = f \to a^T x \leq b$$

Z

f

$f \in \{0, 1\}$

$$a^T x \leq b$$

$$z = 1 - f$$

=

$\geq$

a_j

$$y = f(x)$$

y

x^d

$d + 1$

$$y = exp(x)$$

$a > 0$

$$y = log(x)$$

a

$$x^T Q x + q^T x \leq b$$

$$2x_0^2 + x_0 x_1 + x_1^2$$

$\mathcal{Q}$

$$Bx = b$$

B

b

$$B^T x = b$$

B^T

$$Bx = A_j$$

A_j

A

j

i

$$B^{-1}A$$

$$B^{-1}$$

$expr1$

$expr2$

$$\text{model.addConstr}(expr1 <= expr2)$$
$$\text{model.addConstr}(expr1 == 1)$$
$$\text{model.addConstr}(2 * x + 3 * y <= 4)$$

$qexpr1$

$qexpr2$

$$\text{model.addQConstr}(qexpr1 <= qexpr2)$$
$$\text{model.addQConstr}(qexpr1 == 1)$$
$$\text{model.addQConstr}(2 * x * x + 3 * y * y <= 4)$$

$$\text{model.addConstr}(expr1, GRB.LESS\_EQUAL, expr2)$$
$$\text{model.addConstr}(expr1, GRB.EQUAL, 1)$$

model.addQConstr($qexpr1, GRB.LESS\_EQUAL, qexpr2$)
model.addQConstr($qexpr1, GRB.EQUAL, 1$)

$$\text{model.AddConstr}(expr1 <= expr2)$$
$$\text{model.AddConstr}(expr1 == 1)$$
$$\text{model.AddConstr}(2 * x + 3 * y <= 4)$$

$$\text{model.AddQConstr}(qexpr1 <= qexpr2)$$
$$\text{model.AddQConstr}(qexpr1 == 1)$$
$$\text{model.AddQConstr}(2 * x * x + 3 * y * y <= 4)$$

→

$$Ax = b$$

<

>

$$x'_{Q_L} Q x_{Q_R} + c' x_c = \text{rhs}$$

$$\text{model.addConstr}(qexpr1 <= qexpr2)$$
$$\text{model.addConstr}(qexpr1 == 1)$$
$$\text{model.addConstr}(2 * x * x + 3 * y * y <= 4)$$

$$Ax + b$$

$$2.0 * x$$

$$2.0 * var$$

$$x^T Q x + c^T x + \text{alpha}$$

$$\ell \leq x \leq u$$

$$x^T Qc\, x + q^T x \le \text{beta}$$

x_i

alpha

$$x[\text{resvar}] = \max\{\text{con}, x[j] : j \in \text{vars}\}$$

$-\infty$

$$x[\mathrm{resvar}] = \min\{\mathrm{con}, x[j] : j \in \mathrm{vars}\}$$

∞

$$x[\text{resvar}] = |x[\text{argvar}]|$$

$$x[\text{resvar}] = \text{and}\{x[i] : i \in \text{vars}\}$$

$$x[\text{resvar}] = \text{or}\{x[i] : i \in \text{vars}\}$$

$$x[\text{binvar}] = \text{binval} \Rightarrow \sum \left(x(j) \cdot \mathrm{a}(j)\right) \ \text{ sense rhs}$$

$x[\mathrm{binvar}]$

$$\sum (x[\text{vars}(j)] \cdot \text{val}(j)) \;\; \text{sense rhs}$$

$\leq$

$$x[\mathrm{yvar}] = f(x[\mathrm{xvar}])$$

$$x[\text{yvar}] = p_0 x[\text{xvar}]^d + p_1 x[\text{xvar}]^{d-1} + ... + p_{d-1} x[\text{xvar}] + p_d$$

$$x[\text{yvar}] = \exp(x[\text{xvar}])$$

$$x[\mathrm{yvar}] = \mathrm{a}^{x[\mathrm{xvar}]}$$

$$x[\text{yvar}] = \log(x[\text{xvar}])$$

$$x[\text{yvar}] = \log(x[\text{xvar}]) \setminus \log(a)$$

$$x[\mathrm{yvar}] = x[\mathrm{xvar}]^{\mathrm{a}}$$

$$x[\mathrm{yvar}] = \sin(x[\mathrm{xvar}])$$

$$x[\mathrm{yvar}] = \cos(x[\mathrm{xvar}])$$

$$x[\text{yvar}] = \tan(x[\text{xvar}])$$

$$|ObjBound - ObjVal|/|ObjVal|$$

$ObjBound$

$ObjVal$

$$3x + 4y \leq 5z$$

$$x > y$$

$$x - y$$

$$x <= Mb$$

M

$$3x^2 + 4y^2 + 5z \leq 10$$

$$x^T x \leq y^2$$

$$x^T x \leq yz$$

$f()$

$$r = \max\{x_1, \ldots, x_k, c\}$$

$x_1, \ldots, x_k$

$$(r = 3, x_1 = 2, x_2 = 3, x_3 = 0)$$

$$r = \max\{x_1, x_2, x_3, 1.7\}$$

3

2

1.7

$$r = \min\{x_1, \ldots, x_k, c\}$$

$$(r = 3, x = -3)$$

$$r = \text{and}\{x_1, \ldots, x_k\}$$

$$(r = 1, x_1 = 1, x_2 = 1, x_3 = 1)$$

$$r = \text{and}\{x_1, x_2, x_3\}$$

$$r = \mathrm{or}\{x_1, \ldots, x_k\}$$

$$y = f \to a^T x \leq b$$

$$y \neq f$$

$$y = 1 - f$$

(x, y)

$$(x_1, y_1), (x_2, y_2), ..., (x_n, y_n)$$

$$
\begin{array}{rcll}
r & = & x_j + s_j & \text{for all } j = 1, \ldots, k \\
r & = & c + s_{k+1} & \\
z_1 + \ldots + z_{k+1} & = & 1 & \\
SOS1(s_j, z_j) & & & \text{for all } j = 1, \ldots, k+1 \\
s_j & \geq & 0 & \text{for all } j = 1, \ldots, k+1 \\
z_j & \in & \{0, 1\} & \text{for all } j = 1, \ldots, k+1
\end{array}
$$

$$r \geq \max\{x_1, \ldots, x_k, c\}$$

$s_j \geq 0$

$$r \leq \max\{x_1, \ldots, x_k, c\}$$

$z_j \in \{0, 1\}$

s_j

z_j

$$z_j = 1 \rightarrow s_j = 0$$

$$r = x_j$$

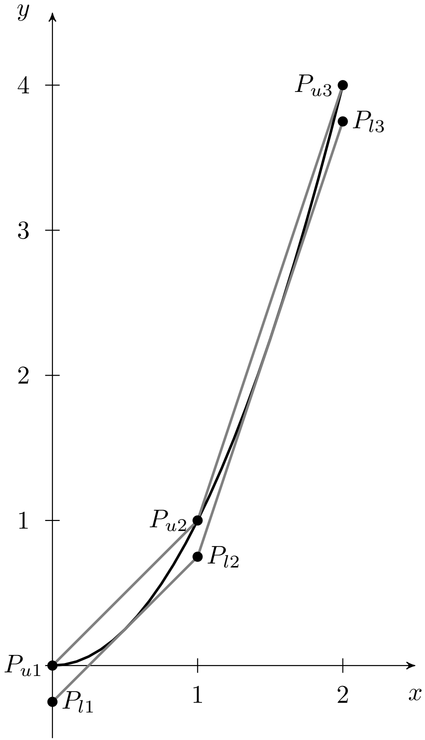
$$r = c$$

$$
\begin{aligned}
r &\geq x_j \quad \text{for all } j = 1, \ldots, k \\
r &\geq c
\end{aligned}
$$

$$y = p_0x^n + p_1x^{n-1} + ... + p_nx + p_{n+1}$$

$$y = ln(x)$$

$$a >= 0$$



y
4
3
2
1
P_{u3}
P_{l3}
P_{u2}
P_{l2}
P_{u1}
P_{l1}
1
2
x

$$y = x^2$$

1.0

$$[0, 2]$$

$P_{u1}(0,0), P_{u2}(1,1), P_{u3}(2,4)$

$P_{l1}(0, -0.25), P_{l2}(1, 0.75), P_{l3}(2, 3.75)$

0.0

P_{l1}, P_{l2}

P_{l3}

P_{u1}, P_{u2}

$$P_{u3}$$

0.6

P_{ui}

0.4

P_{li}

$(0, -0.1), (1, 0.9)$

$(2, 3.9)$

$$-1$$

0.25

$$y = 2x - 1$$

$$(x, y) = (1, 1)$$

$$x = 0.9$$

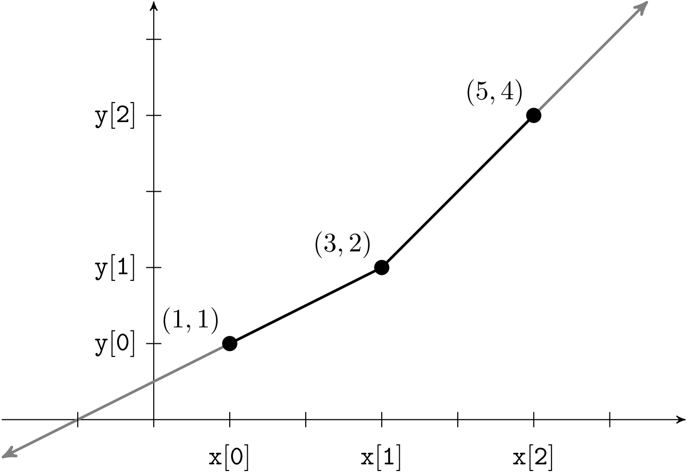
$$x = 1.1$$

1.01

$$x = 1$$

$$3x + 4y + 2$$

$$f(x)$$



(5, 4)
(3, 2)
(1, 1)
y[2]
y[1]
y[0]
x[0]
x[1]
x[2]

$$(1, 1)$$

$$(3, 2)$$

$$(5, 4)$$

$$f(1) = 1$$

$$f(3) = 2$$

$$f(5) = 4$$

$$f(-1) = 0$$

$$f(6) = 5$$

$$\mathtt{x} = [x_1, \ldots, x_n], \quad \mathtt{y} = [y_1, \ldots, y_n]$$

$$f(v) = \begin{cases} y_1 + \frac{y_2 - y_1}{x_2 - x_1}(v - x_1), & \text{if } v \leq x_1, \\ y_i + \frac{y_{i+1} - y_i}{x_{i+1} - x_i}(v - x_i), & \text{if } v \geq x_i \text{ and } v \leq x_{i+1}, \\ y_n + \frac{y_n - y_{n-1}}{x_n - x_{n-1}}(v - x_n), & \text{if } v \geq x_n. \end{cases}$$

$$x = x_i$$

$$(x_{i-1}, y_{i-1}), (x_i, y_i), (x_{i+1}, y_{i+1}), (x_{i+2}, y_{i+2})$$

$$x_i = x_{i+1}$$

$$y_i \neq y_{i+1}$$

$$x_{i-1} \leq x < x_i$$

(x_{i-1}, y_{i-1})

(x_i, y_i)

$$x_i \leq x < x_{i+2}$$

$$(x_{i+1}, y_{i+1})$$

$$(x_{i+2}, y_{i+2})$$

y_i

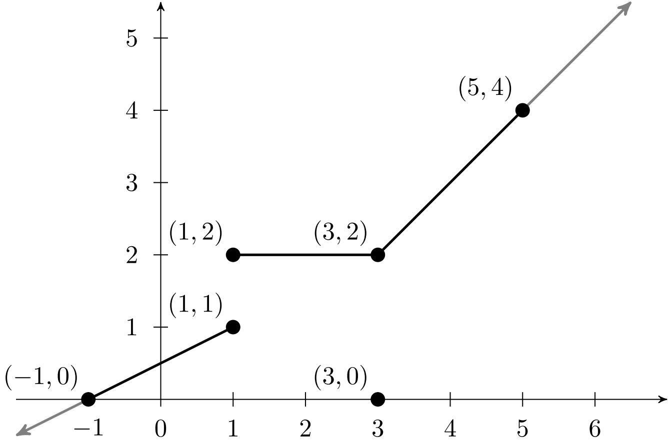
$$y_{i+1}$$

$$(x_{i-2}, y_{i-2}), (x_{i-1}, y_{i-1}), (x_i, y_i), (x_{i+1}, y_{i+1}), (x_{i+2}, y_{i+2})$$

$$x_{i-1} = x_i = x_{i+1}$$

$$y_i \neq y_{i-1}$$

$$y_{i-1}$$

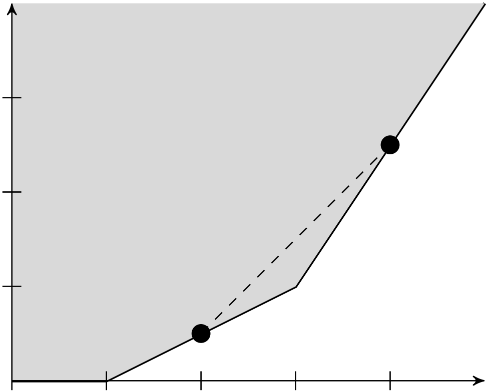


5
4
3
2
1
(5, 4)
(1, 2)
(3, 2)
(1, 1)
(−1, 0)
(3, 0)
−1
0
1
2
3
4
5
6

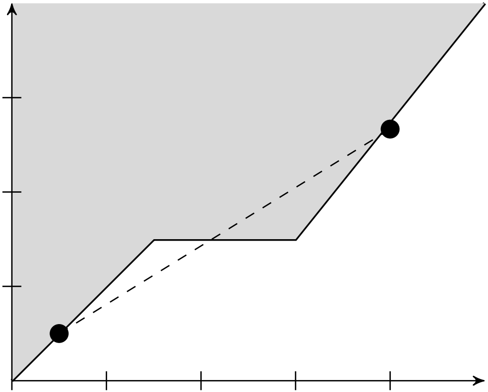
$$(-1,0),(1,1),(1,2),(3,2),(3,0),(3,2)$$

$(1, 2)$

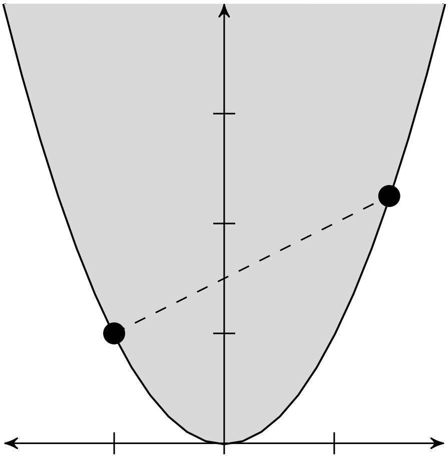
$(3, 0)$



$$f(1)$$



$$3x^2 + 4y^2 + 2xy + 2x + 3$$



$1e-6$

k

$$\bar{a}x = \lambda^t A x \le \lambda^t b = -\beta + \sum_{j:\bar{a}_j<0} \bar{a}_j U_j + \sum_{j:\bar{a}_j>0} \bar{a}_j L_j,$$

$\beta > 0$

L_j

U_j

$\lambda_i \geq 0$

$\lambda_i \leq 0$

$\bar{a}_j \geq 0$

$$U_j = \infty$$

$$\bar{a}_j \leq 0$$

$$L_j = -\infty$$

β

λ

$$\begin{array}{ll} \text{minimize} & c'x \\ \text{subject to} & Ax \geq b \\ & x \geq 0 \end{array}$$

$$
\begin{array}{ll}
\text{maximize} & b'y \\
\text{subject to} & A'y \leq c \\
& y \geq 0
\end{array}
$$

$\geq$

$$\geq 0$$

$\leq$

≤ 0

$$-2$$

$$a^T x \leq b$$

$$a^T x + s = b$$

$$a^T x + s$$

l

q

S

g

$$l + q + s$$

$$l + q + s + g$$

$$i - l - q - s$$

$$a^T y \geq c$$

$$a^T y - z = c$$

$$a^T y - z$$

$$|z|$$

m

$cols$

set

X

Set

$m.NumVars$

$m.numvars$

$$2^{31} - 1$$

$$y \le Mb$$

$1e-5$

Mb

z_P

z_D

$$gap = |z_P - z_D| / |z_P|$$

$$z_P = z_D = 0$$

$$z_P = 0$$

$$z_D \neq 0$$

$$\left|z_P - z_D\right|$$

$$(10X01^2 + 2X01 * X02 + 2X02 * X01 + 2X02^2)/2$$

$$10X01^2$$

$$4X01 * X02$$

$$2X02^2$$

$$
\begin{array}{ll}
\text{minimize} & y - 1.3x(1-z) + (1-z) \\
\text{subject to} & 2y - 3x + 1.7w = 1.7 \\
& -y + x + xz(1-v) \geq 0 \\
& -y \leq 0, \\
& v, w, x, y, z \in \{0, 1\}.
\end{array}
$$

$$AA^T$$

$$1 + x + 2y$$

$$y + 2z$$

$$-1 \cdot (1 + x + 2y) + 2 \cdot (y + 2z) = -1 - x + 4z$$

$$10^6$$

10

5

100

$$\text{base\_value} = \max\{bestsol, bestbound + |bestbound| * rgap, bestbound + agap\},$$

$bestsol$

bestbound

$rgap$

$agap$

20

$$\{2, 2, 1\}$$

$$\{0.10, 0.05, 0.00\}$$

$$\{0, 1, 2\}$$

$$\max\{10 \cdot 0.10, 10 \cdot 0.05, 0, 1\} \ = \ 1$$

$$\begin{aligned} x - 6y &= 1 \\ 0.333x - 2y &= .333 \end{aligned}$$

$$y := 0.1665x - 0.1665$$

$$\begin{aligned} x - 6 \cdot (0.1665x - 0.1665) &= 1 \\ \Leftrightarrow 0.001x &= 0.001 \end{aligned}$$

$$x = 1, \ y = 0$$

$$\begin{aligned} x - 6y &= 1 \\ 0.3333333333333333x - 2y &= 0.3333333333333333 \end{aligned}$$

$$y := 0.1666666666666667x - 0.1666666666666667$$

$$\begin{aligned} x - 6 \cdot (0.1666666666666667x - 0.1666666666666667) &= 1 \\ \Leftrightarrow 2 \cdot 10^{-16}x + 1 + 2 \cdot 10^{-16} &\approx 1 \end{aligned}$$

$$x = 6y + 1$$

$$1 \approx 1 + 2 \cdot 10^{-16}$$

$\leq$

$$a \cdot x \leq b$$

$$a \cdot y \leq c$$

$$
\begin{array}{lll}
\min & 0 & \\
s.t. & x \leq & 0 \\
 & x \geq & 10^{-10}
\end{array}
$$

10^{-6}

10^{-5}

10^3

10^{-9}

$x \in [-10^{-6}, 10^{-6}]$

$$x \in [-10^{10}, 10^{10}]$$

10^{-16}

$$(P) \max\{cx : Ax = b, l \leq x \leq u\}$$

D

$$D_{ii} > 0, \; \forall i$$

$$(P)$$

$$(P_D)$$

$$(P_D)\max\{cDx' : ADx' = b, D^{-1}l \leq x' \leq D^{-1}u\}$$

10^4

$$10^{10}$$

P

$$f_1(x)$$

$f_2(x)$

$$\overline{f}(x) = Mf_1(x) + f_2(x)$$

10^{-13}

$$
\begin{aligned}
10^{-7}x + 10y &\leq 10 \\
x + 10^4 z &\leq 10^3 \\
x, y, z &\geq 0,
\end{aligned}
$$

$$[10^{-7}, 10^4]$$

$$10^5$$

$$\begin{aligned} 10^{-2}x' + 10y &\leq 10 \\ 10^{2}x' + 10z &\leq 1 \\ x', y, z &\geq 0, \end{aligned}$$

$$x = 10^5 x'$$

$$[10^{-2}, 10^{2}]$$

$$[10^{-3}, 10^{6}]$$

$$\begin{aligned} x - 10^6 y &\geq 0 \\ y &\in [0, 10] \end{aligned}$$

$$
\begin{array}{rcl}
x - 10y_1 & \geq & 0 \\
y_1 - 10y_2 & = & 0 \\
y_2 - 10y_3 & = & 0 \\
y_3 - 10y_4 & = & 0 \\
y_4 - 10y_5 & = & 0 \\
y_5 - 10y & = & 0 \\
y & \in & [0, 10]
\end{array}
$$

$$y = -10^{-6}, \ x = -1$$

$$
\begin{aligned}
x - 10^3 y' &\geq 0 \\
y' &\in [0, 10^4]
\end{aligned}
$$

$$10^{-3}y' = y$$

-10^{-3}

-10^{-9}

$$
\begin{aligned}
x &\leq 10^6 y \\
x &\geq 0 \\
y &\in \{0,1\},
\end{aligned}
$$

$$y = 0.0000099999$$

9.999

$$
\begin{aligned}
x &\leq 10^3 y \\
x &\geq 0 \\
y &\in \{0,1\}
\end{aligned}
$$

$x \leq 0.01$

$$y = 0 \Rightarrow x = 0$$

10^9

$$6 \cdot 10^6/0.00099 = 6.0606 \cdot 10^9.$$

10^{12}

10^{-8}

0.1

10^{-17}

$$A^{-1}b$$

$\mathcal{E}$

$$A^{-1}(b+\varepsilon)$$

$$\eta(b, \varepsilon) := \frac{\|A^{-1}b\|}{\|A^{-1}(b+\varepsilon)\|} \Big/ \frac{\|b\|}{\|b+\varepsilon\|}.$$

$$\kappa(A) := \max_{b,\varepsilon} \eta(b,\varepsilon).$$

$$\kappa(A) = \frac{\lambda_{\max}}{\lambda_{\min}},$$

$\lambda_{\max}$

$\lambda_{\min}$

$$\kappa(A) = \|A\| \|A^{-1}\|.$$

$$\kappa(A) = 10^k$$

κ

$$\begin{array}{ll} \max & cx \\ s.t. & Ax \leq b. \end{array}$$

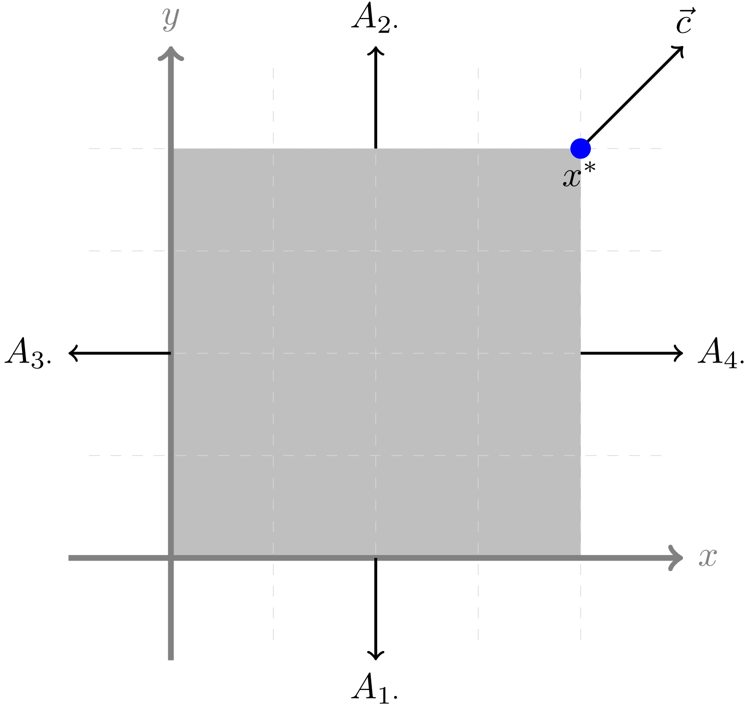
$$
\begin{array}{llrl}
\max & x+y & \vec{c} = & (1,1) \\
s.t. & -x \leq 0 & A_{1\cdot} = & (-1,0) \\
 & x \leq 1 & A_{2\cdot} = & (1,0) \\
 & -y \leq 0 & A_{3\cdot} = & (0,-1) \\
 & y \leq 1 & A_{4\cdot} = & (0,1).
\end{array}
$$

$$b^t := (0, 1, 0, 1)$$

$\max_{x \in \mathbb{R}^2} \{\vec{c}x : Ax \le b\}.$

$\vec{c}$

x^*

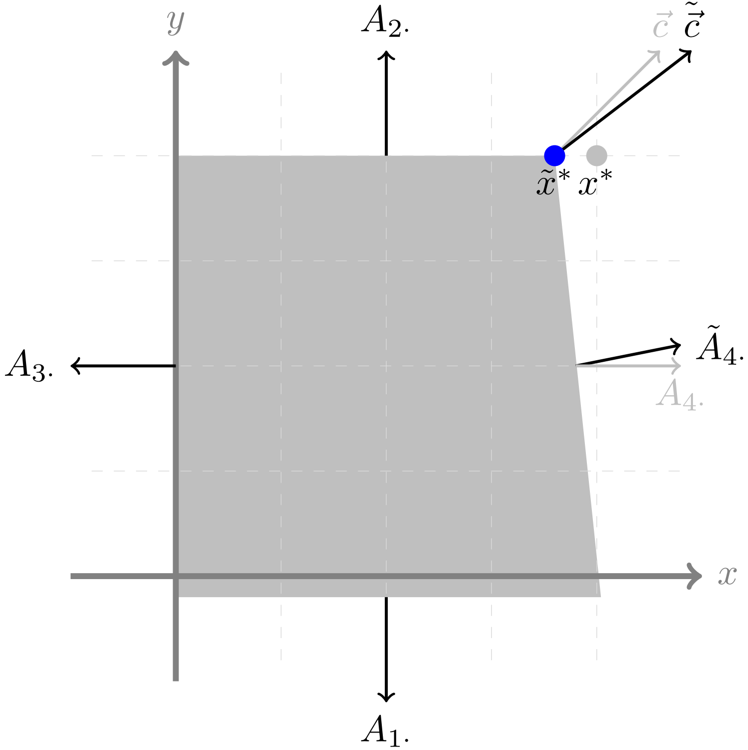

y
$A_2.$
$\vec{c}$
x^*
$A_3.$
$A_4.$
x
$A_1.$

$$\vec{c}x$$

$$\tilde{b}^t = (\varepsilon, 1, 0, 1)$$

$$\tilde{\vec{c}} = (1 + \varepsilon, 1)$$

$$\tilde{A}_4. = (\varepsilon, 1)$$



y
$A_{2\cdot}$
$\vec{c}$ $\tilde{\vec{c}}$
$\tilde{x}^*$ x^*
$A_{3\cdot}$
$\tilde{A}_{4\cdot}$
$A_{4\cdot}$
x
$A_{1\cdot}$

$A_4.$

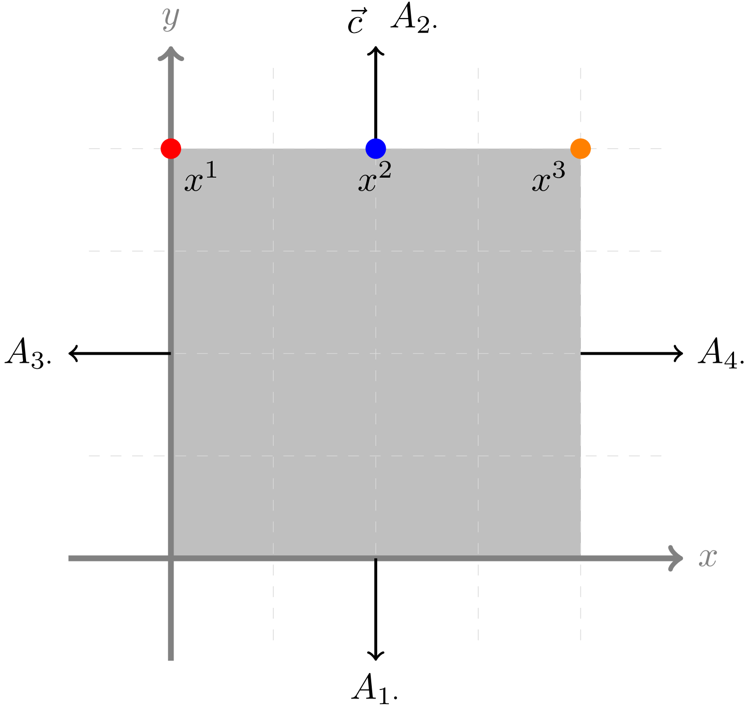
$x \leq 1$

$$100x \leq 100$$

$$x + \varepsilon y \leq 1$$

$$100x + \varepsilon y \leq 100$$

$$
\begin{array}{lrrl}
\max & y & \vec{c} = & (0,1) \\
s.t. & -x \leq 0 & A_{1\cdot} = & (-1,0) \\
 & x \leq 1 & A_{2\cdot} = & (1,0) \\
 & -y \leq 0 & A_{3\cdot} = & (0,-1) \\
 & y \leq 1 & A_{4\cdot} = & (0,1).
\end{array}
$$



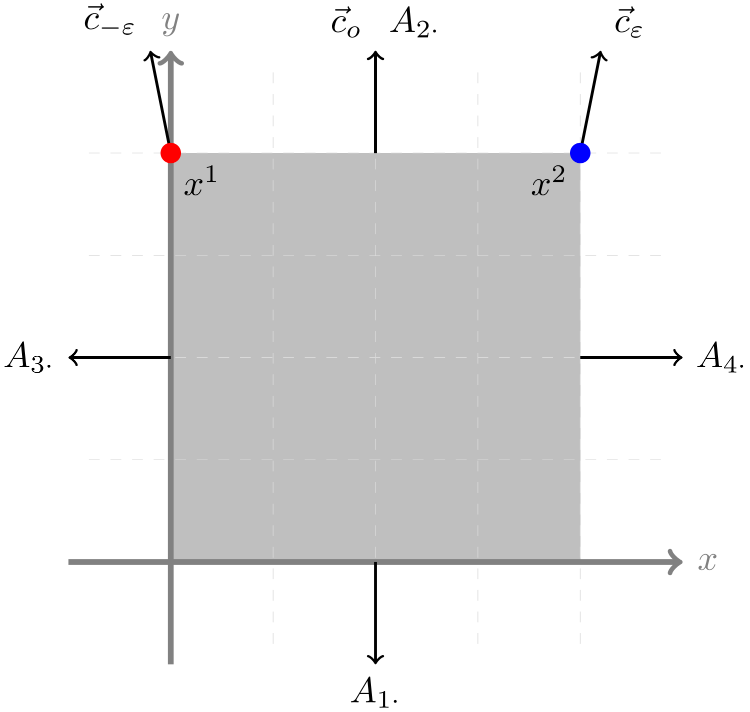
y
$\vec{c}$
$A_2.$
x^1
x^2
x^3
$A_3.$
$A_4.$
x
$A_1.$

$$x^1$$

$$x^3$$

x^2

$$\begin{array}{lrrl} \max & \varepsilon x + y & \vec{c} = & (\varepsilon, 1) \\ s.t. & -x \leq 0 & A_{1\cdot} = & (-1, 0) \\ & x \leq 1 & A_{2\cdot} = & (1, 0) \\ & -y \leq 0 & A_{3\cdot} = & (0, -1) \\ & y \leq 1 & A_{4\cdot} = & (0, 1). \end{array}$$



$\vec{c}_{-\varepsilon}$
y
$\vec{c}_o$
$A_{2\cdot}$
$\vec{c}_{\varepsilon}$
x^1
x^2
$A_{3\cdot}$
$A_{4\cdot}$
x
$A_{1\cdot}$

$$(0, 0)$$

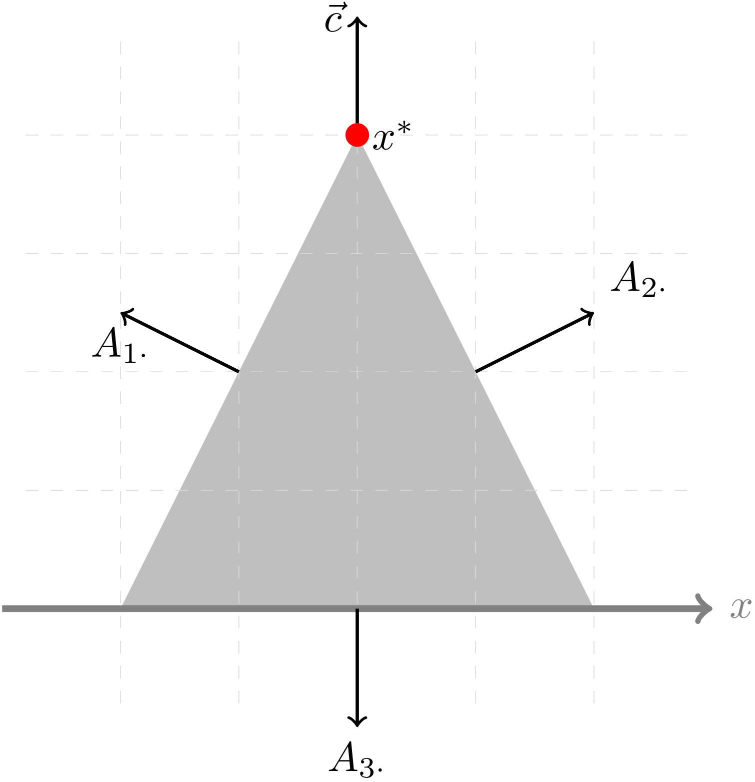
$$(0, 1)$$

$(10^6, 1)$

$$(10^6, 0)$$

$$1 - 10^6\varepsilon$$

$$
\begin{array}{llrl}
\max & y & \vec{c} = & (0,1) \\
s.t. & -x + \varepsilon y \leq 1 & A_{1\cdot} = & (-1, \varepsilon) \\
 & x + \varepsilon y \leq 1 & A_{2\cdot} = & (1, \varepsilon) \\
 & -y \leq 0 & A_{3\cdot} = & (0, -1)
\end{array}
$$


$\vec{c}$
x^*
$A_{2\cdot}$
$A_{1\cdot}$
x
$A_{3\cdot}$

$A_1.$

$A_2.$

$$x^* = (0, \tfrac{1}{\varepsilon})$$

$$(1 + \delta, 1)$$

$$\tilde{x}^* = \left(-\frac{\delta}{2}, \frac{2+\delta}{2\varepsilon}\right)$$

L_1

$$\|x^* - \tilde{x}^*\|_1 = \frac{|\delta|}{2} + \frac{|\delta|}{\varepsilon}$$

δ

$$(-1, \delta)$$

$$\tilde{x}^* = (1 - \frac{2\varepsilon}{\varepsilon+\delta}, \frac{2}{\varepsilon+\delta})$$

$$\delta = \varepsilon/2$$

$$\frac{1}{\varepsilon}$$

$$\frac{4}{3\varepsilon}$$

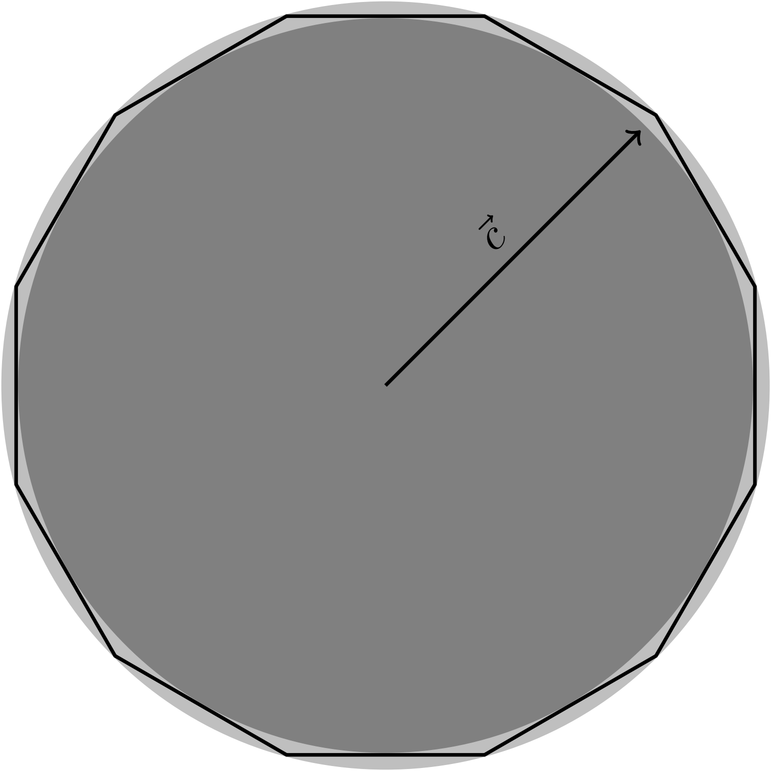
$$\lim_{\varepsilon \to 0^+} \|x^*\| = \infty$$

$$y \leq 10^4$$

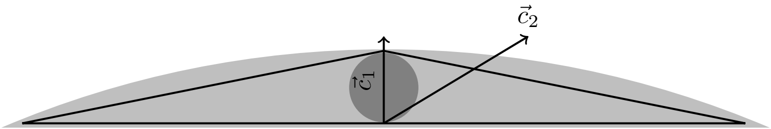
10^{-4}

$$\sin(2\pi \tfrac{i}{10^6})x + \cos(2\pi \tfrac{i}{10^6})y \le 1,\ \forall i \in \{1, \ldots, 10^6\},$$

$\mathbb{R}^2$


$\vec{\mathfrak{c}}$

$[-1, 1]$



$\vec{c}_2$
$\vec{c}_1$

$\vec{c}_1$

$$\vec{c}_1 = (0, 1)$$

$$\vec{c}_2 = (1, 0)$$